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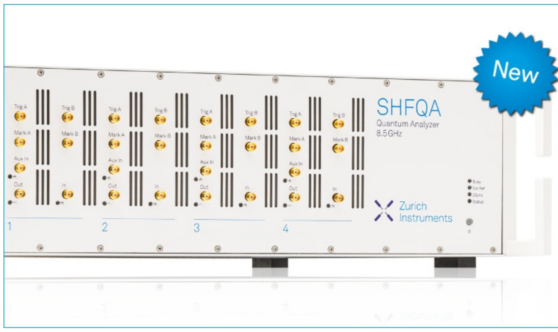

Unidirectional Marangoni–Poiseuille Flows of a Viscous Incompressible Fluid with the Navier Boundary Condition

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Abstract. The paper obtains an exact solution describing the shear convection of a swirling viscous incompressible fluid in a horizontal layer taking into account the Marangoni condition, the Navier condition and the non-uniform pressure distribution on one of the layer boundaries. It is shown that this exact solution is able to describe the appearance of the stratification of a velocity field and thermal force fields.

INTRODUCTION

It is known [1] that the motion of a viscous Newtonian fluid is described by the Navier–Stokes equations. Finding exact solutions to these equations is currently of interest in the context of modeling convective processes in a fluid. Convective processes caused by linear temperature distribution at the boundary were first described by Ostroumov [2] and Birikh [3]. A more general class of solutions – a class of solutions that are linear by a part of horizontal coordinates, but as applied to problems of magnetic hydrodynamics – was proposed by Lin [4]. Later, this class was adapted for convection [5, 6] and thermal diffusion [7]. In addition to the problem of class construction, the choice of boundary conditions is a separate problem, since it is known that the solution of partial differential equations strongly depends on these conditions. Traditionally, the no-slip condition is used in the study of fluid flows along a solid surface. However, experiments [8] show that this condition is often violated. This paper examines the influence of the Navier slip condition [9] and of the nonzero pressure gradient on the features of the velocity field, temperature field, and pressure field topologies during fluid flow in a flat horizontal layer [10–16].

BOUNDARY VALUE PROBLEM FORMULATION

In the case of steady-state motion of a viscous incompressible fluid in the flat case, the system of heat convection equations [1] takes the form

$$V_x \frac{\partial V_x}{\partial x} = -\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial z^2} \right); \; \frac{\partial P}{\partial z} = g\beta T \; ; \; V_x \frac{\partial T}{\partial x} = \chi \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right); \; \frac{\partial V_x}{\partial x} = 0 \, . \tag{1}$$

Here, $V_x = V_x(x,z)$ is the x-component of the velocity vector $\boldsymbol{V}$; $P(x,z)$ is deviation of pressure from hydrostatic, divided by average density; $T(x,z)$ is deviation of temperature from the reference value; ν, χ are the kinematic (molecular) viscosity of the fluid and the coefficient of thermal diffusivity; β is the temperature coefficient of volumetric expansion of the fluid; g is acceleration of gravity.

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System (1) is overdetermined. One approach to solving such systems is to choose a special class of solutions for which redundant equations are satisfied identically. As such a class, we consider the following [10, 11] one:

$$V_x = U(z);\ V_y = V_z = 0; \tag{2}$$

$$P = P_0(z) + xP_1(z);\ T = T_0(z) + xT_1(z). \tag{3}$$

By substituting the class (2)–(3) into system (1) and applying the method of undetermined coefficients, it can be shown that system (1) becomes equivalent to the following system of ordinary differential equations:

$$T_1'' = 0;\ P_1' = g\beta T_1;\ \nu U'' = P_1;\ \chi T_0'' = UT_1;\ P_0' = g\beta T_0. \tag{4}$$

In system (4), the prime denotes derivation with respect to the vertical variable *z*. In the integration of this eighth-order system there appear eight integration constants whose values are to be determined from the corresponding number of boundary conditions.

We assume that the lower boundary $z = 0$ is thermally insulated and that, at the upper boundary $z = h$, the source of thermal disturbance is specified. Taking into account the structure of class (2)–(3), these conditions are written in the form

$$T_0(0) = T_1(0) = 0;\ T_0(h) = 0;\ T_1(h) = A. \tag{5}$$

In addition, we assume that nonuniform pressure distribution is set at the upper boundary as

$$P_0(h) = S_0;\ P_1(h) = S_1. \tag{6}$$

We also assume that the velocity is determined from the slip condition [9] specified on the lower boundary and from the Marangoni condition [17] specified on the upper boundary,

$$\alpha \left.\frac{\partial U}{\partial z}\right|_{z=0} = U(0);\ \eta \left.\frac{\partial U}{\partial z}\right|_{z=h} = -\sigma T_1(h). \tag{7}$$

Here, σ and η are the coefficients of temperature-induced surface tension and dynamic viscosity, respectively; α is the slip length.

EQUATION SYSTEM SOLUTION

The integration of system (4), in view of the system of boundary conditions (5)–(7), yields the following polynomial exact solution:

$$T_1 = \frac{Az}{h};\ P_1 = \frac{Ag\beta}{2h}\left(z^2 - h^2\right);$$

$$U = \frac{S_1}{2\nu}\left[z^2 - 2h(z+\alpha)\right] + \frac{Ag\beta}{24\nu h}\left[z^4 - 6h^2z^2 + 8h^3(z+\alpha)\right] - \frac{A\sigma}{\eta}(z+\alpha);$$

$$T_0 = \frac{AS_1}{120h\nu\chi}\left[3z^5 - 10hz^4 - 20hz^3\alpha + h^3z(7h+20\alpha)\right] +$$

$$+\frac{A^2g\beta}{5040h^2\nu\chi}\left[5z^7 - 63h^2z^5 + 140h^3z^4 + 280h^3z^3\alpha - 2h^5z(41h+140\alpha)\right] +$$

$$+\frac{A^2\sigma}{12h\chi\eta}(h-z)z\left[h^2 + (h+z)(z+2\alpha)\right];$$

$$P_0 = S_0 - \frac{g\beta A S_1}{240h\nu\chi}(z-h)^2\left(4h^4 - z^4 + 2hz^2(z+5\alpha) + 2h^3(4z+5\alpha) + 5h^2 z(z+4\alpha)\right) -$$
$$-\frac{g\beta\sigma A^2}{120h\eta\chi}(z-h)^2\left(3h^3 + 2hz(2z+5\alpha) + z^2(2z+5\alpha) + h^2(6z+5\alpha)\right) +$$
$$+\frac{A^2 g^2 \beta^2}{40320h^2\nu\chi}(z-h)^2\left[183h^6 - 69h^2 z^4 + 10hz^5 + 5z^6 +\right.$$
$$\left. + h^4 z(221z + 1120\alpha) + h^5(366z + 560\alpha) + 4h^3 z^2(19z + 140\alpha)\right]. \qquad (8)$$

It is easy to see that each component of the exact solution (8) is determined by the interaction of three flows: the flow caused by the pressure drop (Poiseuille flow); the flow caused by the heating/cooling of the boundaries and the action of the gravity force (thermogravitational flow); the flow caused by the heating/cooling of the boundaries and the effect of the surface tension of the fluid (thermocapillary flow). The overlap of these flows greatly complicates the topology of the hydrodynamic fields.

INVESTIGATION OF THE SOLUTION

Note that, when setting boundary value problems, one often ignores the longitudinal pressure gradient. Taking into account the pressure gradient S_1 in the exact solution (8) leads to a more complex structure of the solution and to an increase in the number of critical points

For example, for some values of the parameter S_1, the longitudinal gradient P_1 can vanish once inside the layer $[0, h]$; consequently, the pressure field $P_1 x$ can have one stratification point, whereas it can have none when $S_1 = 0$. Another illustrative example is the behavior of velocity in the absence of heat sources at the boundary. In the case that both boundaries are thermally insulated (the longitudinal temperature gradient A is zero), the velocity is described by the expression

$$U = \frac{S_1}{2\nu}\left[z^2 - 2h(z+\alpha)\right],$$

and it is nonzero, although it has no stagnant points in the layer $[0, h]$.

As a result of studying the behavior of polynomial (8), which determines the velocity U, it has been found that, when the temperature gradient A is nonzero, the velocity U can have two zero points in the layer only if the fluid is abnormal ($\sigma < 0$), Fig. 1. Otherwise, the velocity U can have only one stagnant point in the layer.

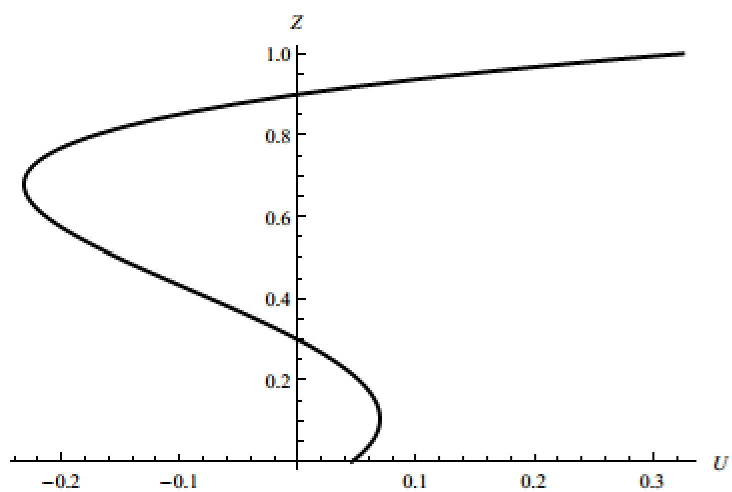

FIGURE 1. The behavior of the velocity

The analysis of the spectral properties of the polynomials determining the exact solution (8) has shown that, in the general case (when the longitudinal gradients A and S_1 are nonzero), each hydrodynamic field can have several stratification points. When both gradients are zero ($S_1 \equiv 0$, $A \equiv 0$), the solution obviously degenerates into trivial, T_1=P_1=U=T_0=0, P_0=S_0; therefore, it cannot describe field stratification.

CONCLUSION

An exact solution describing the Marangoni flow of a viscous incompressible fluid in a flat horizontal layer has been proposed. A distinctive feature of the constructed solution (in addition to taking into account the slip condition on the solid boundary) is the allowance for the non-uniformity of pressure distribution on the upper surface of the layer in question. It has been shown that hydrodynamic fields can have several stratification points.

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